Morse Theory

Reference Book: «Morse Theory», J. Milnor

Part I

Non-degenerate Smooth Functions on a Manifold

1 Definitions and Lemmas

$f \in C^\infty(M)$ $p \in M$: critical point

[$f_*: T_pM \to T_{f(p)}R \cong R$ is zero]

$M^a \triangleq \{x \in M \mid f(x) \leq a\}$

non-degenerate: $\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \middle| p\right)$ is non-singular

The Hessian of f at p:

$v, w \in T_pM$ $\tilde{v}, \tilde{w}$ are extended vector fields

then define $f_{**}(v, w) = \tilde{v}_p(\tilde{w}(f))$

f_{**} is symmetric and well defined

If $v=\sum a_i \frac{\partial}{\partial x_i}\Big|_p$ $w=\sum b_j \frac{\partial}{\partial x_j}\Big|_p$ then $\tilde{w}=\sum_j b_j \frac{\partial}{\partial x_j}\Big|_p$ ②

take to be a constant function

$$f_{**}(v,w)=v\left(\sum b_j \frac{\partial f}{\partial x_j}\right)=\sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}(p)$$

The index of f_{**}: the maximal dimension of a subspace where f_{**} is negative definite.

nullity: the dimension of null space, the space consisted of v s.t. $f_{**}(v,w)=0$ $\forall w\in T_pM$

Lemma 1.1 $f\in C^\infty(V)$ V is a convex nbhd of 0 $f(0)=0$. Then $f(x_1,\cdots x_n)=\sum_{i=1}^n x_i g_i(x_1,\cdots x_n)$. for some C^∞ functions g_i, with $g_i(0)=\frac{\partial f}{\partial x_i}(0)$.

Lemma 1.2 (Morse) p is a non-degenerate critical point of f Then $\exists$ a local coordinate $(y^1,\cdots y^n)$ in a nbhd U of p with $y^i(p)=0$ $(\forall i)$, s.t (λ is the index at p)

$$f=f(p)-(y^1)^2-\cdots(y^\lambda)^2+(y^{\lambda+1})^2+\cdots(y^n)^2.$$

Cor 1.3 Non-degenerate critical points are isolated

1-parameter group of diffeomorphisms:

C^∞ map $\varphi: \mathbb{R}\times M \to M$, s.t.

(i) $\forall t\in\mathbb{R}$ $\varphi_t: M\to M$, $q\mapsto \varphi(t,q)$ is a diffeomorphism

(ii) $\forall t,s\in\mathbb{R}$ $\varphi_{t+s}=\varphi_t\circ\varphi_s$

The vector field <u>generates</u> φ: $X_q(f)=\lim_{h\to 0}\frac{f(\varphi_h(q))-f(q)}{h}$

Lemma 1.4. A compact supported smooth vector field generates a unique 1-parameter group of diffeomorphisms

pf. Consider a curve $t\mapsto c(t)$, define its <u>velocity vector</u> $\frac{dc}{dt}\in T_{c(t)}M$ by $\frac{dc}{dt}(f)=\lim_{h\to 0}\frac{f(c(t+h))-f(c(t))}{h}=\frac{d(f\circ c)}{dt}$

Then the group φ generated by X satisfies

$$\frac{d\varphi_t(q)}{dt}=X_{\varphi_t(q)} \quad \text{[curve } t\mapsto\varphi_t(q) \text{ for fixed } q]$$

Then $\forall q\in M$ $\exists U_q\ni q$ and $\varepsilon>0$

s.t. $\frac{d\varphi_t(q)}{dt}=X_{\varphi_t(q)}$ has a unique smooth solution on U, $|t|<\varepsilon$

Then cover the support with finitely many those neighborhoods. Details omitted here.

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2 Homotopy Type and Critical Values

Thm 2.1 $f\in C^\infty(M)$ $a<b$, suppose $f^{-1}([a,b])$ is compact and contains no critical points. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b.

[Choose a Riemannian metric on M. Then the gradient of f is characterized by

$$\langle X, \operatorname{grad} f\rangle = X(f)$$

In local notation, $\operatorname{grad} f = \left(\sum_{j=1}^{n} g^{ij}\frac{\partial f}{\partial x_j}\right)\partial_i$.

thus it vanishes precisely at the critical point of f

If $c: \mathbb{R}\to M$ is a curve, then

$$\left\langle \frac{dc}{dt}, \operatorname{grad} f\right\rangle = \frac{d(f\circ c)}{dt}$$]

pf. Let $\rho: M\to\mathbb{R}$ be a smooth function which equals $\langle \operatorname{grad} f, \operatorname{grad} f\rangle$ on $f^{-1}([a,b])$ and vanishes outside a compact neighborhood of $f^{-1}([a,b])$

Define $X_q = \rho(q)(\operatorname{grad} f)_q$

Then by Lemma 1.4, it generates $\{\varphi_t : M \to M\}$

For fixed $q \in M$, if $\varphi_t(q) \in f^{-1}([a,b])$,

then $\frac{d f(\varphi_t(q))}{dt} = \langle \frac{d\varphi_t(q)}{dt}, \operatorname{grad} f \rangle = \langle X, \operatorname{grad} f \rangle = 1$

Then $\varphi_{b-a} : M \to M$ is a diffeomorphism between M^a and M^b

Define $r_t : M^b \to M^b$ $(0 \le t \le 1)$

$$q \mapsto \begin{cases} q, & f(q) \le a \\ \varphi_{t(a-f(q))}(q) & a \le f(q) \le b \end{cases}$$

$r_0 = Id$

r_1: retraction

Then r_t is the deformation retract. #

Thm 2.2 $f \in C^\infty(M)$, p is a non-degenerate critical point with index λ and $f(p) = c$

Suppose $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact and contains only one critical point p, for some $\varepsilon > 0$

Then for all sufficiently small ε,

$M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with λ-cell attached.

Pf. Choose a coordinate system $u^1, \cdots, u^n$ in a nbhd U of p, with

$$f = c - (u^1)^2 - \cdots - (u^\lambda)^2 + (u^{\lambda+1})^2 + \cdots + (u^n)^2 \text{ in } U.$$

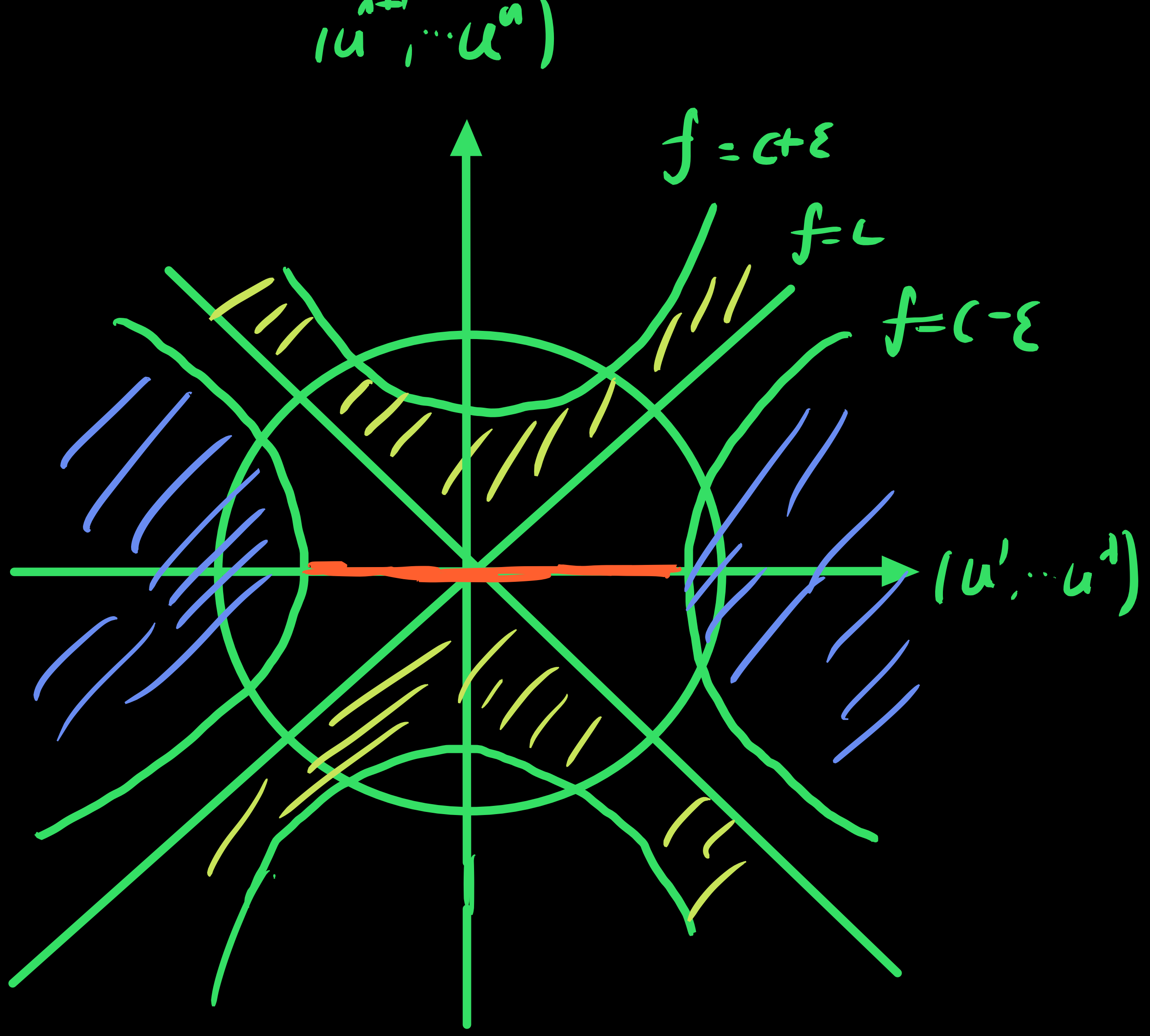
And $u^1(p)=\cdots=u^n(p)=0$. Choose ε small, s.t. ⑥

(1) $f^{-1}([c-\varepsilon, c+\varepsilon])$ is small and have no critical points other than p

(2) The image of u under $(u^1,\cdots u^n): u\to\mathbb{R}^n$ contains the closed ball $\{(u^1,\cdots u^n) \mid \sum_{i=1}^{n}(u^i)^2\le 2\varepsilon\}$

Define $e^\lambda=\{q\in u \mid (u^1(q))^2+\cdots(u^\lambda(q))^2\le\varepsilon,\ u^{\lambda+1}(q)=\cdots u^n(q)=0\}$

Note that $\partial e^\lambda=e^\lambda\cap M^{c-\varepsilon}$, so it remains to show $M^{c-\varepsilon}\cup e^\lambda$ is a deformation retract of $M^{c+\varepsilon}$



/// : $M^{c-\varepsilon}$

/// : $f^{-1}([c,c+\varepsilon])$

— : e^λ

Let $\mu \in C^\infty(\mathbb{R})$ s.t. $\begin{cases} \mu(0) > \varepsilon \\ \mu(r) = 0 & r \geq 2\varepsilon \\ -1 < \mu'(r) \leq 0 & \forall r \in \mathbb{R} \end{cases}$

$F \in \underline{C^\infty(M)}$ [check] s.t. $F = \begin{cases} f - \mu\left((u^1)^2 + \cdots (u^\lambda)^2 + 2(u^{\lambda+1})^2 + \cdots (u^n)^2\right) & \text{in } U \\ f & \text{otherwise} \end{cases}$

Then define $\xi, \eta : U \to [0, \infty)$

$$\xi = (u^1)^2 + \cdots (u^\lambda)^2 \quad \eta = (u^{\lambda+1})^2 + \cdots (u^n)^2$$

Step1: $F^{-1}((-\infty, c+\varepsilon]) = f^{-1}((-\infty, c+\varepsilon])$

[Outside of $\{\xi + 2\eta \leq 2\varepsilon\}$ $f = F$

In $\{\xi + 2\eta \leq 2\varepsilon\}$ $F \leq f = c - \ \ + \eta \leq c + \frac{1}{2}\xi + \eta$
$\leq c + \varepsilon$]

Step 2: The critical points of F are the same as f

[$\frac{\partial F}{\partial \xi} = -1 - \mu'(\xi + 2\eta) < 0 \quad \frac{\partial F}{\partial \eta} = 1 - 2\mu'(\xi + 2\eta) \geq 1$

$dF = \frac{\partial F}{\partial \xi} d\xi + \frac{\partial F}{\partial \eta} d\eta$

$d\xi, d\eta$ are simultaneously 0 only at the origin.

thus F has no critical points in U other than p

In $F^{-1}([c-\varepsilon, c+\varepsilon])$, by Step 1 and $F \leq f$

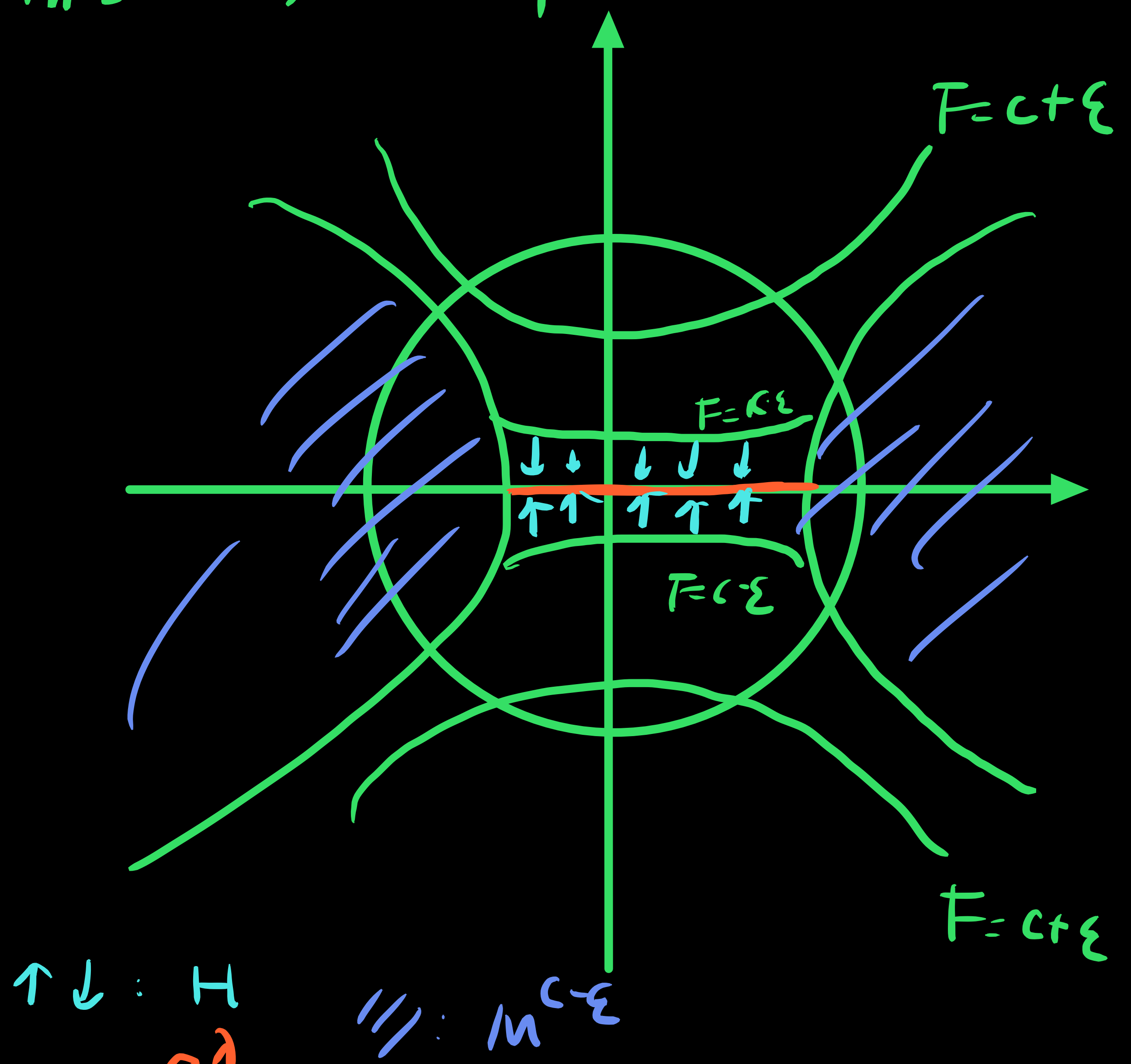
we have $F^{-1}([c-\varepsilon, c+\varepsilon]) \subset f^{-1}([c-\varepsilon, c+\varepsilon])$, thus

compact

It can contain no critical points of F except ⑧
possibly p. But $F(p) = c - \mu(0) < c - \varepsilon$
Thus $F^{-1}([c-\varepsilon, c+\varepsilon])$ contains no critical points]

Step 3: $F^{-1}((-\infty, c-\varepsilon])$ is a deformation retract of $M^{c+\varepsilon}$.

[This is from Step 1, 2 and Thm 2.1]



Let $H = \overline{F^{-1}((-\infty, c-\varepsilon]) - M^{c-\varepsilon}}$ then $F^{-1}((-\infty, c-\varepsilon]) = M^{c-\varepsilon} \cup H$

Note that $\forall q \in e^\lambda$. $\xi(q) \leq \varepsilon$, $\eta(q) = 0$. then $e^\lambda \subset H$

Since $\frac{\partial F}{\partial \xi} < 0$. $F(q) \leq F(p) < c - \varepsilon$. but $f(q) \geq c - \varepsilon$

Step 4: $M^{c-\varepsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\varepsilon} \cup H$ ⑨

[Define $r_t : M^{c-\varepsilon} \cup H \to M^{c-\varepsilon} \cup H$ by following:

(i) Outside of U $r_t = Id$

(ii) $\xi \le \varepsilon$ $(u^1, \cdots u^n) \mapsto (u^1, \cdots u^\lambda, tu^{\lambda+1}, \cdots tu^n)$

(iii) $\varepsilon \le \xi \le \eta + \varepsilon$ $(u^1, \cdots u^n) \mapsto (u^1, \cdots u^\lambda, s_t u^{\lambda+1}, \cdots s_t u^n)$

$$s_t = t + (1-t)\sqrt{\frac{\xi - \varepsilon}{\eta}}$$

(iv) $\eta + \varepsilon \le \xi$ $r_t = Id$]

Combining the 4 steps completes the proof #.

Rmk 2.3 If there are k non-degenerate critical points $P_1 \cdots P_k$ with indices $\lambda_1, \cdots \lambda_k$ in $f^{-1}(c)$. Then $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon} \cup e^{\lambda_1} \cdots \cup e^{\lambda_k}$.

Thm 2.4 $f \in C^\infty(M)$ with no degenerate critical points and if each M^a is compact, then M has the homotopy type of a CW-complex, with one cell of dim λ for each critical point of index λ

3. Examples

Thm 3.1 (Reeb) M is a compact manifold. $f \in C^\infty(M)$ with only two non-degenerate critical points, then M is homeomorphic to a sphere.

Pf. Say that $f(p)=0$ is the minimum, $f(q)=1$ is the maximum. If ε is small enough then $M^\varepsilon = f^{-1}([0,\varepsilon])$ and $f^{-1}([1-\varepsilon,1])$ are closed n-cells (1.2)

By Thm 2.1. M^ε is homeomorphic to $M^{1-\varepsilon}$.

Then M is the union of two closed n-cells, matched along their common boundary. thus M is homeomorphic to S^n.

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[The result holds even if the critical points degenerate. And M doesn't have to be diffeomorphic to S^n]

Rmk 3.2 If a function on M^n has three non-degenerate critical points, then by Poincaré's duality, they have index $0, n, \frac{n}{2}$. And M^n has the homotopy type of an $\frac{n}{2}$-sphere with an n-cell attached.

Ex $f: \mathbb{C}P^n \to \mathbb{R}$.

$(z_0 : z_1 \cdots z_n) \mapsto \sum c_j |z_j|^2$ $\quad c_0, \cdots c_n$ are distinct real constants

$U_0 = \{(z_0 : z_1 \cdots : z_n) \mid z_0 \neq 0\}$. $\quad |z_0| \frac{z_j}{z_0} = x_j + i y_j$.

then $\varphi: U_0 \to \mathbb{R}^{2n}$

$(z_0 ; \cdots z_n) \mapsto (x_1, y_1, \cdots x_n, y_n)$ is a coordinate map.

$$f = c_0 + \sum_{j=1}^{n} (c_j - c_0)(x_j^2 + y_j^2)$$

Then the only critical point of f within U_0 is $(1 : 0, \cdots : 0)$ which is non-degenerate. The index is twice the number of j which $c_j < c_0$

Then consider $U_1, \cdots U_n$ similarly.

By Thm 2.4 $\mathbb{C}P^n$ has the same homotopy type of a CW-complex of the form

$$e^0 \cup e^2 \cup e^4 \cdots \cup e^{2n}.$$

Thus $H_i(\mathbb{C}P^n ; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, 2, \cdots 2n \\ 0, & \text{otherwise} \end{cases}$

4. The Morse Inequalities

Def Let S be a function from certain pairs of spaces to integers. S is <u>subadditive</u> if whenever $X \supset Y \supset Z$, $S(X,Z) \leq S(X,Y) + S(Y,Z)$.

If equality holds, S is called <u>additive</u>.

Lemma 4.1 S is subadditive. $X_0 \subset \cdots \subset X_n$.

Then $S(X_n, X_0) \leq \sum_{i=1}^{n} S(X_i, X_{i-1})$. If S is additive, the equality holds.

Thm 4.2 (Weak Morse Inequality) C_λ denotes the number of critical points of index λ on the compact manifold M, then

$$R_\lambda(M) \leq C_\lambda$$

$$\sum (-1)^\lambda R_\lambda(M) = \sum (-1)^\lambda C_\lambda$$

R_λ: Betti number of (X,Y) [subadditive]

pf. Let $a_1 < \cdots < a_k$ be such that M^{a_i} contains exactly i critical points, and $M^{a_k} = M$. Then

$$H_*(M^{a_i}, M^{a_{i-1}}) = H_*(M^{a_{i-1}} \cup e^{\lambda_i}, M^{a_{i-1}})$$

$$= H_*(e^{\lambda_i}, \dot{e}^{\lambda_i}) \quad \text{[Excision]}$$

By Lemma 4.1 with $S = R_\lambda \Rightarrow R_\lambda(M) \leq \sum_{i=1}^{k} R_\lambda(M^{a_i}, M^{a_{i-1}}) = C_\lambda$

$S = \chi \Rightarrow \chi(M) = \sum_{i=1}^{k} \chi(M^{a_i}, M^{a_{i-1}}) = C_0 - C_1 \cdots \pm C_n$ #

Lemma 4.3 $S_\lambda(X,Y) = R_\lambda(X,Y) - R_{\lambda-1}(X,Y) \cdots \pm R_0(X,Y)$ is subadditive.

[pf is omitted here]

Then we obtain the Morse Inequalities

$$S_\lambda(M) \le \sum_{i=1}^{k} S_\lambda(M^{a_i}, M^{a_{i-1}}) = C_\lambda - C_{\lambda-1} \cdots \pm C_0$$

or $R_\lambda(M) - R_{\lambda-1}(M) \cdots \pm R_0(M) \le C_\lambda - C_{\lambda-1} \cdots \pm C_0$

This is sharper than the previous weak inequalites

Cor 4.4 If $C_{\lambda+1} = C_{\lambda-1} = 0$, then $R_\lambda = C_\lambda$ $R_{\lambda+1} = R_{\lambda-1} = 0$

5. Manifolds in Euclidean Space

Let $M \subset \mathbb{R}^n$ be a k-dim manifold.

$N = \{(q,v) \mid q \in M,\ v \perp M \text{ at } q\} \subset M \times \mathbb{R}^n$

N is an n-dim manifold embedded in $\mathbb{R}^{2n}$

$E: N \to \mathbb{R}^n, \quad (q,v) \mapsto q+v$

Def. $e \in \mathbb{R}^n$ is a focal point of (M,q) with multiplicity μ

If $e = q+v$, $(q,v) \in N$, and $JE|_{(q,v)}$ has nullity $\mu > 0$

Thm 5.1 (Sard) M_1, M_2 are smooth manifolds with same dimension $f: M_1 \to M_2$ is C^1. then the image of the critical points has the measure 0 in M_2

Cor 5.2 Almost all $x \in \mathbb{R}^n$ is not a focal point of M.

Pf. x is a focal point iff x is a critical value of $E: N \to \mathbb{R}^n$

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$u^1, \cdots u^k$ is the basis of M locally. $\vec{x}(u^1, \cdots u^k)$ is the embedding to $\mathbb{R}^n$.

The first fundamental form: $(g_{ij}) = \left(\frac{\partial \vec{x}}{\partial u_i} \cdot \frac{\partial \vec{x}}{\partial u_j}\right)$

Second fundamental form: $(\vec{l}_{ij}) = ($ the normal component of $\frac{\partial^2 \vec{x}}{\partial u^i \partial u^j})$

WLOG, we can assume $(g_{ij})_{\vec{q}} = I$

Then the eigenvalues of $(\vec{v}\cdot \vec{\ell}_{ij})$ are the principle curvatures $k_1, \cdots k_k$ of M at $\vec{q}$ in the normal direction $\vec{v}$

Consider the normal line $l = \vec{q} + t\vec{v}$, $\vec{v}$ a fixed unit orthogonal vector.

Lemma 5.3. The focal points of $(M, \vec{q})$ along l are precisely the points $\vec{q} + k_i^{-1}\vec{v}$, where $1 \le i \le k$, $k_i \neq 0$. Thus there are at most k focal points along l.

Pf Choose $n-k$ vector fields $\vec{w}_1(u^1, \cdots u^k), \cdots \vec{w}_{n-k}(u^1, \cdots u^k)$ s.t. $\vec{w}_1, \cdots \vec{w}_{n-k}$ are unit vectors which are orthogonal to each other and to M.

Introduce coordinates $(u^1, \cdots u^k, t^1, \cdots t^{n-k})$ on N.

$$(u^1, \cdots u^k, t^1, \cdots t^{n-k}) \longleftrightarrow \left(\vec{x}(u^1, \cdots u^k), \sum_{\alpha=1}^{n-k} t^\alpha \vec{w}_\alpha(u^1, \cdots u^k)\right) \in N$$

Then in the coordinates $E: N \to \mathbb{R}^n$

$$(u^1, \cdots u^k, t^1, \cdots t^{n-k}) \xrightarrow{\vec{e}} \vec{x}(u^1, \cdots u^k) + \sum t^\alpha \vec{w}_\alpha(u^1, \cdots u^k)$$

with
$$\begin{cases} \dfrac{\partial \vec{e}}{\partial u^i} = \dfrac{\partial \vec{x}}{\partial u^i} + \sum_\alpha t^\alpha \dfrac{\partial \vec{w}_\alpha}{\partial u^i} \\ \dfrac{\partial \vec{e}}{\partial t^\beta} = \vec{w}_\beta \end{cases}$$

Take inner products of these vectors with linear independent vectors $\frac{\partial \vec{x}}{\partial u^1}, \cdots \frac{\partial \vec{x}}{\partial u^k}, \vec{w}_1, \cdots \vec{w}_{n-k}$

we obtain an $n\times n$ matrix whose rank equals the rank of $J\varepsilon$.

$$A=\begin{pmatrix} \left(\frac{\partial \vec{x}}{\partial u^i}\cdot\frac{\partial \vec{x}}{\partial u^j}+\sum_\alpha t^\alpha \frac{\partial \vec{w}_\alpha}{\partial u^i}\cdot\frac{\partial \vec{x}}{\partial u_j}\right) & \left(\sum_\alpha t^\alpha \frac{\partial \vec{w}_\alpha}{\partial u^i}\cdot\vec{w}_\beta\right) \\ 0 & \mathrm{Id} \end{pmatrix}$$

Then $\mathrm{null}(A)=\mathrm{null}\left(\frac{\partial \vec{x}}{\partial u^i}\cdot\frac{\partial \vec{x}}{\partial u^j}+\sum_\alpha t^\alpha \frac{\partial \vec{w}_\alpha}{\partial u^i}\cdot\frac{\partial \vec{x}}{\partial u^j}\right)$

$=\mathrm{null}\left(g_{ij}-\sum_\alpha t^\alpha \vec{u}^\alpha\cdot\vec{\ell}_{ij}\right)$

thus $\vec{q}+t\vec{v}$ is a focal point with multiplicity μ

iff $(g_{ij}-t\vec{v}\cdot\vec{\ell}_{ij})$ is singular with nullity μ.

iff $\frac{1}{t}$ is an eigenvalue of $(\vec{v}\cdot\vec{\ell}_{ij})$ with multiplicity μ

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For a fixed $\vec{p}\in\mathbb{R}^n$.

$L_{\vec{p}}=f: M\to\mathbb{R}$

$\vec{x}(u^1,\cdots u^k)=\vec{x}\cdot\vec{x}-2\vec{x}\cdot\vec{p}+\vec{p}\cdot\vec{p}$

then $\frac{\partial f}{\partial u^i}=2\frac{\partial \vec{x}}{\partial u^i}(\vec{x}-\vec{p})$

thus f is a critical point at $\vec{q}$ iff $\vec{q}-\vec{p}$ is normal to M.

$$\frac{\partial^2 f}{\partial u^i\partial u^j}=2\left(\frac{\partial \vec{x}}{\partial u^i}\cdot\frac{\partial \vec{x}}{\partial u^j}+\frac{\partial^2 \vec{x}}{\partial u^i\partial u^j}(\vec{x}-\vec{p})\right)$$

$$\overset{\vec{p}=\vec{x}+t\vec{v}}{=\!=}\ 2(g_{ij}-t\vec{v}\cdot\vec{\ell}_{ij})$$

Lemma 5.4 $\vec{q}\in M$ is a degenerate critical point of $f=L_{\vec{p}}$ iff $\vec{p}$ is a focal point of $(M,\vec{q})$

Thm 5.5 For almost all $\vec{p} \in \mathbb{R}^n$ (except for measure 0) $L_p : M \to \mathbb{R}$ has no degenerate critical points

Cor 5.6 On any manifold M, $\exists$ a differentiable function with no degenerate critical points for which each M^a is compact

Lemma 5.7 The index of $L_{\vec{p}}$ at a non-degenerate critical point $\vec{q} \in M$ is equal to the number of focal points of (M, q) which lie on the segment from q to p, counted with multiplicity.

pf. The index of the matrix

$$\frac{\partial^2 L_{\vec{p}}}{\partial u^i \partial u^j} = 2(g_{ij} - t\vec{v} \cdot \vec{\ell}_{ij})$$

is equal to the number of negative eigenvalues

$\Leftrightarrow$ the number of eigenvalues of $(\vec{v} \cdot \vec{\ell}_{ij})$ which are $\geq \frac{1}{t}$. Then the result follows from Lemma 5.3.

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6 The Lefschetz Theorem on Hyperplane Sections

Part II

A Rapid Course In Riemannian Geometry

7 Covariant Differentiation

Def An affine connection at a point $p \in M$ is a function which assigns to each $X_p \in T_pM$ and to each vector field Y a new vector

$$\nabla_{X_p} Y \in T_pM$$

called the covariant derivative of Y in the direction X_p.

And we require: $\forall f: M \to \mathbb{R}$

$$\nabla_{X_p}(fY) = (\nabla_{X_p} f) Y + f(p) \nabla_{X_p} Y$$
$$= (X_p f) Y + f(p) \nabla_{X_p} Y$$

Global affine connection. $\nabla_X Y$ is bilinear

$$\nabla_{fX} Y = f \nabla_X Y$$
$$\nabla_X (fY) = f \nabla_X Y + (Xf) Y$$

∂_i the basis of T_pM. $\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k$

$$X = \sum X^i \partial_i \quad Y = \sum Y^i \partial_i$$
$$\Rightarrow \nabla_X Y = \sum_k \left(\sum_i \left(\partial_i Y^k + \sum_j \Gamma_{ij}^k Y^j \right) X^i \right) \partial_k$$

Given a curve $c: \mathbb{R} \to M$, any vector field V along c determines its covariant derivative $\frac{DV}{dt}$. (20)

① $\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$ ② $\frac{D(fV)}{dt} = \frac{df}{dt}V + f\frac{DV}{dt}$

③ If $V_t = Y_{c(t)}$ for each t. $Y \in \Gamma^\infty(TM)$,

then $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y$

Lemma 7.1 Uniqueness of covariant derivative

pf. $V = \sum v^j \partial_j$ then $\frac{DV}{dt} = \sum_j \frac{dv^j}{dt}\partial_j + v^j \nabla_{\frac{dc}{dt}} \partial_j$

$$= \sum_k \left(\frac{dv^k}{dt} + \sum_{i,j} v^j \frac{du^i}{dt} \Gamma^k_{ij} \right) \partial_k$$

Check that this satisfies ①②③. #

V is parallel if $\frac{DV}{dt} = 0$

Lemma 7.2 Given c. $V_0 \in T_{c(0)}M$. $\exists!$ parallel vector field V along c with $V_{c(0)} = V_0$.

Def. A connection ∇ on M is compatible with the metric if $\forall$ curve c and $\forall P, P'$ parallel along c, $\langle P, P' \rangle$ is constant.

Lemma 7.3 If ∇ is compatible. V, W are any two vector fields along c, $\frac{d}{dt}\langle V, W \rangle = \langle \frac{DV}{dt}, W \rangle + \langle V, \frac{DW}{dt} \rangle$

pf. Choose parallel orthonormal basis $P_1, \cdots P_n$

$v = \sum v^i P_i \quad w = \sum w^j P_j \quad \langle v, w \rangle = \sum v^i w^i$

$$\frac{Dv}{dt} = \sum \frac{dv^i}{dt} P_i \qquad \frac{Dw}{dt} = \sum \frac{dw^j}{dt} P_j$$

$$\langle \frac{Dv}{dt}, w \rangle + \langle v, \frac{Dw}{dt} \rangle = \sum \left(\frac{dv^i}{dt} w^i + v^i \frac{dw^i}{dt} \right) = \frac{d}{dt} \langle v, w \rangle$$ #

Cor 7.4 For any vector fields Y, Y' on M. $\forall X_p \in T_pM$

$$X_p(\langle Y, Y' \rangle) = \langle \nabla_{X_p} Y, Y'_p \rangle + \langle \nabla_{X_p} Y', Y_p \rangle$$

Def A connection ∇ is symmetric if $\nabla_X Y - \nabla_Y X = XY - YX$

Take $X = \partial_i, Y = \partial_j \Rightarrow \Gamma_{ij}^k = \Gamma_{ji}^k$

Lemma 7.5 A Riemannian manifold possesses only one symmetric and compatible connection

pf. By Cor 7.4 $\partial_i g_{jk} = \langle \nabla_{\partial_i} \partial_j, \partial_k \rangle + \langle \partial_j, \nabla_{\partial_i} \partial_k \rangle$

Permuting i, j, k. use the symmetric condition

$$\Rightarrow \quad \Gamma_{ij}^l = \sum_k \frac{1}{2} (g_{jk,i} + g_{ik,j} - g_{ij,k}) g^{kl} \quad (*)$$

Check that the connection defined by $(*)$ satisfies the condition

#

Consider a parametrized surface in M. that is a smooth function $s: \mathbb{R}^2 \to M$.

A vector field along s is a assignment to each

$(x,y)\in\mathbb{R}^2$ a vector $V_{(x,y)}\in T_{s(x,y)}M$

$\forall y_0$, restrict V to the curve $x\mapsto s(x,y_0)$,

its covariant derivative w.r.t. x is $\left(\frac{DV}{\partial x}\right)_{(x,y_0)}$

Lemma 7.6 If ∇ is symmetric, then $\frac{D}{\partial x}\frac{\partial s}{\partial y}=\frac{D}{\partial y}\frac{\partial s}{\partial x}$

8. The Curvature Tensor

$$R(X,Y)Z=-\nabla_X\nabla_Y Z+\nabla_Y\nabla_X Z+\nabla_{[X,Y]}Z$$

Lemma 8.1 $(R(X,Y)Z)_p$ only depends on X_p, Y_p, Z_p.

Furthermore, $T_pM\times T_pM\times T_pM\to T_pM$

$(X_p,Y_p,Z_p)\mapsto R(X_p,Y_p)Z_p$ is trilinear

Consider a parametrized surface $s:\mathbb{R}^2\to M$

Lemma 8.2 $\frac{D}{\partial y}\frac{D}{\partial x}V-\frac{D}{\partial x}\frac{D}{\partial y}V=R\left(\frac{\partial s}{\partial x},\frac{\partial s}{\partial y}\right)V$

Henceforth we will assume M is a Riemannian manifold with the unique symmetric and compatible connection.

Lemma 8.3 (1) $R(X,Y)Z+R(Y,X)Z=0$
(2) $R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0$
(3) $\langle R(X,Y)Z,W\rangle+\langle R(X,Y)W,Z\rangle=0$
(4) $\langle R(X,Y)Z,W\rangle=\langle R(Z,W)X,Y\rangle$

9. Geodesics and Completeness

Def $\gamma: I\to M$ is called a geodesic if $\frac{D}{dt}\frac{d\gamma}{dt}=0$

Thus $\frac{d}{dt}\langle\frac{d\gamma}{dt},\frac{d\gamma}{dt}\rangle=2\langle\frac{D}{dt}\frac{d\gamma}{dt},\frac{d\gamma}{dt}\rangle=0$

$\|\frac{d\gamma}{dt}\|$ is constant.

$\gamma(t)=(u^1(t),\cdots u^n(t))$, then

$$\frac{d^2u^k}{dt^2}+\sum_{i,j}^{n}\Gamma_{ij}^k\frac{du^i}{dt}\frac{du^j}{dt}=0$$

Thm 9.1 $\exists$ a nbhd W of $(\vec{u}_1,\vec{v}_1)$ and $\varepsilon>0$, s.t. $\forall(\vec{u}_0,\vec{v}_0)\in W$,

$$\begin{cases}\frac{d^2\vec{u}}{dt^2}=\vec{F}(\vec{u},\frac{d\vec{u}}{dt})\\ \vec{u}(0)=\vec{u}_0\quad \frac{d\vec{u}}{dt}(0)=\vec{v}_0\end{cases}$$

has a solution in $(-\varepsilon,\varepsilon)$

Furthermore, $W\times(-\varepsilon,\varepsilon)\to\mathbb{R}^n$, $(\vec{u}_0,\vec{v}_0,t)\mapsto u(t)$ is C^1

Lemma 9.2 $\forall p_0\in M$ $\exists$ a nbhd U of p, $\varepsilon>0$

s.t. $\forall p\in U$, $v\in T_pM$ with $|v|<\varepsilon$,

$\exists!$ geodesic $\gamma_v:(-2,2)\to M$ with $\gamma_v(0)=p$, $\frac{d\gamma_v}{dt}(0)=v$

$v \in T_qM$. geodesic $\gamma: [0,1] \to M$
$\gamma(0) = q$ $\frac{d\gamma}{dt}(0) = v$

Then $\exp_q(v) \triangleq \gamma(1)$ is called the exponential of v.

Thus $\gamma(t) = \exp_q(tv)$. $\exp_q(v)$ is defined when $\|v\|$ is small. In other words, $\forall p \in M$, $\exists$ a nbhd of $(p,0)$ V in TM. s.t. $(q,v) \mapsto \exp_q(v)$ is defined on V

$F: V \to M \times M$
$(q,v) \mapsto (q, \exp_q(v))$ $\quad J_{F}|_{(p,0)} = \begin{pmatrix} I & I \\ & I \end{pmatrix}$

thus F maps some nbhd V' of $(p,0) \in TM$ diffeomorphically onto some nbhd of (p,p) in $M \times M$

WLOG, assume V' assists of all (q,v), s.t. q belongs to a given nbhd U' of p and $\|v\| < \varepsilon$.

Choose a smaller nbhd W of p s.t. $F(V') \supset W \times W$

Lemma 9.3 $\forall p \in M$. $\exists$ a nbhd W and $\varepsilon > 0$, s.t.

(1) $\forall q_1, q_2 \in M$ are joined by a unique geodesic with length $< \varepsilon$

(2) The geodesic depends smoothly on the two points

(3) $\forall q \in W$ $\exp_q$ maps the open ε-ball in T_qM diffeomorphically onto an open set $U_q \supset W$.

Thm 9.4 Let W and ε be in Lemma 9.3.
$\gamma: [0,1] \to M$ be a geodesic connecting two points with length $< \varepsilon$. $\omega: [0,1] \to M$ be any other smooth path joining them. then

$$\int_0^1 \left\|\frac{d\gamma}{dt}\right\| dt \le \int_0^1 \left\|\frac{d\omega}{dt}\right\| dt$$

with equality holds iff $\omega([0,1]) = \gamma([0,1])$

Cor 9.5 $\omega: [0,1] \to M$, arc length parametrized, has length less than or equal to the length of any other path from $\omega(0)$ to $\omega(1)$. then ω is a geodesic.

<u>Def</u> A geodesic $\gamma: [a,b] \to M$ is minimal if its length is less than or equal to the length of any other piecewise smooth joining its ending points.

Thm 9.6 (Hopf-Rinow)

M is geodesically complete

$\Rightarrow$ { any two points can be joined by a minimal geodesic
every bounded subset of M has compact closure
M is complete as a metric space.

PART III

The Calculus of Variation Applied to Geodesics

10. The path space of smooth manifold

The set of all piecewise smooth path from p to q is denoted by $\Omega(M; p, q)$. (Ω)

The tangent space of Ω at a path ω will be meant the vector space consisting of all piecewise smooth vector fields W along ω, with $W(0)=W(1)=0$

<u>Def</u> A variation of ω is a function $\bar{\alpha}: (-\varepsilon, \varepsilon) \to \Omega$ for some $\varepsilon > 0$. s.t.

(1) $\bar{\alpha}(0) = \omega$ (2) $\alpha: (-\varepsilon, \varepsilon) \times [0,1] \to M$, $\alpha(u,t) = \bar{\alpha}(u)(t)$ is piecewise smooth

(3) $\alpha(u,0) = p$ $\alpha(u,1) = q$. $\forall u \in (-\varepsilon, \varepsilon)$

If $(-\varepsilon, \varepsilon)$ replaced by a nbhd of 0 in $\mathbb{R}^n$, then α (or $\bar{\alpha}$) is called an n-parameter variation of ω.

A vector field W along ω given by

$$W_t = \frac{\partial \alpha}{\partial u}(0,t) =: \frac{d\bar{\alpha}}{du}(0)_t$$

$$W(0)=W(1)=0 \quad \exists\, W \in T_\omega \Omega$$

W is the variation field of α

Given any $W \in T_\omega \Omega$, $\exists$ a variation

$$\bar{\alpha} : (-\varepsilon, \varepsilon) \to \Omega \quad \text{satisfying} \quad \bar{\alpha}(0)=\omega, \quad \frac{d\bar{\alpha}}{du}(0) = W$$

Set $\bar{\alpha}(u)(t) = \exp_{\omega(t)}(uW_t)$

If F is a real function on Ω.

given $W \in T_\omega \Omega$, choose a variation $\bar{\alpha} : (-\varepsilon, \varepsilon) \to \Omega$

satisfying $\bar{\alpha}(0)=\omega$, $\frac{d\bar{\alpha}}{du}(0) = W$

Set $F_*(W)$ equal to $\left.\frac{d(F(\bar{\alpha}(u)))}{du}\right|_{u=0}$ multiplied by

$\left(\frac{d}{dt}\right)_{F(\omega)}$. $\quad F_* : T_\omega \Omega \to T_{F(\omega)} R \cong R$

Def ω is a critical path for a function $F: \Omega \to R$

iff $\left.\frac{dF(\bar{\alpha}(u))}{du}\right|_{u=0}$ is zero for every variation $\bar{\alpha}$ of ω.

11. Energy of a Path

$\forall \omega \in \Omega$, define the energy of ω from a to b

as $E_a^b(\omega) = \int_a^b \|\frac{d\omega}{dt}\|^2 dt$. $\qquad E \triangleq E_0^1$

$L_a^b(\omega) = \int_a^b \|\frac{d\omega}{dt}\| dt \xrightarrow{\text{Cauchy}} (L_a^b)^2 \leq (b-a) E_a^b$

with equality holds $\Leftrightarrow$ t is proportional to arc length

Suppose $\exists$ a minimal geodesic γ from $p = \omega(0)$ to $q = \omega(1)$.

then $E(\gamma) = L(\gamma)^2 \leq L(\omega)^2 \leq E(\omega)$

"=" holds iff ω is a minimal geodesic on length parametrized

Lemma 11.1. Let M be a complete Riemannian manifold.

$p, q \in M$ have distance d. then

$$E : \Omega(M, p, q) \to \mathbb{R}$$

take on its minimal d^2 precisely on the set of minimal geodesics from p to q.

$\bar{\alpha} : (-\varepsilon, \varepsilon) \to \Omega$ be a variation of ω. $W_t = \frac{\partial \bar{\alpha}}{\partial u}(0, t)$ Let

$V_t = \frac{d\omega}{dt}$ velocity field.

$A_t = \frac{D}{dt}\frac{d\omega}{dt}$: acceleration field.

$\Delta_t V = V_{t^+} - V_{t^-}$: discontinuity

Thm 11.2 (First Variation)

$$\frac{1}{2}\frac{dE(\bar{\alpha}(u))}{du}\Big|_{u=0} = -\sum_t \langle W_t, \Delta_t V \rangle - \int_0^1 \langle W_t, A_t \rangle dt$$

Pf $\frac{\partial}{\partial u}\langle \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}\rangle = 2\langle \frac{D}{du}\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}\rangle$

Therefore $\frac{dE(\tilde{\alpha}(u))}{du} = \frac{d}{du}\int_0^1 \langle \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}\rangle dt$

$$= 2\int_0^1 \langle \frac{D}{\partial u}\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}\rangle dt$$

$$= 2\int_0^1 \langle \frac{D}{\partial t}\frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial t}\rangle dt$$

$$\frac{\partial}{\partial t}\langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial t}\rangle = \langle \frac{D}{\partial t}\frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial t}\rangle + \langle \frac{\partial \alpha}{\partial u}, \frac{D}{\partial t}\frac{\partial \alpha}{\partial t}\rangle$$

$$\Rightarrow \int_{t_{i-1}}^{t_i} \langle \frac{D}{\partial t}\frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial t}\rangle dt = \langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial t}\rangle \Big|_{t_{i-1}^+}^{t_i^-} - \int_{t_{i-1}}^{t_i} \langle \frac{\partial \alpha}{\partial u}, \frac{D}{\partial t}\frac{\partial \alpha}{\partial t}\rangle dt$$

$$\Rightarrow \frac{1}{2}\frac{dE(\tilde{\alpha}(u))}{du} = -\sum_{i=1}^{k-1} \langle \frac{\partial \alpha}{\partial u}, \Delta_{t_i}\frac{\partial \alpha}{\partial t}\rangle - \int_0^1 \langle \frac{\partial \alpha}{\partial u}, \frac{D}{\partial t}\frac{\partial \alpha}{\partial t}\rangle dt$$

$$\Rightarrow \frac{1}{2}\frac{dE(\tilde{\alpha}(u))}{du}\Big|_{u=0} = -\sum_t \langle W, \Delta_t V\rangle - \int_0^1 \langle W, A\rangle dt$$

#

Cor 11.3 The path ω is a critical point for the function E iff ω is a geodesic.

12. The Hessian of the Energy Function

$f \in C^\infty(M)$ with critical point p, then

$f_{**}: T_pM \times T_pM \to \mathbb{R}$ is the Hessian as follows:

$X_1, X_2 \in T_pM$, choose a map $(u_1, u_2) \mapsto \alpha(u_1, u_2)$ on a nbhd of $(0,0)$ in $\mathbb{R}^2$ ($\alpha(u_1,u_2) \in M$)

s.t. $\alpha(0,0) = p$, $\frac{\partial \alpha}{\partial u_1}(0,0) = X_1$, $\frac{\partial \alpha}{\partial u_2}(0,0) = X_2$.

then $f_{**}(X_1, X_2) \triangleq \left.\frac{\partial^2 f(\alpha(u_1,u_2))}{\partial u_1 \partial u_2}\right|_{(0,0)}$

Now given $W_1, W_2 \in T_\gamma \Omega$. γ a geodesic. choose a

2-parameter variation $\alpha: U \times [0,1] \to M$, U a nbhd of $(0,0)$ in $\mathbb{R}^2$

s.t. $\alpha(0,0,t) = \gamma(t)$ $\frac{\partial \alpha}{\partial u_1}(0,0,t) = W_1(t)$ $\frac{\partial \alpha}{\partial u_2}(0,0,t) = W_2(t)$

Then define $E_{**}: T_\gamma\Omega \times T_\gamma\Omega \to \mathbb{R}$

$$(W_1, W_2) \mapsto \left.\frac{\partial^2 E(\tilde{\alpha}(u_1,u_2))}{\partial u_1 \partial u_2}\right|_{(0,0)} =: \frac{\partial^2 E}{\partial u_1 \partial u_2}(0,0)$$

Thm 12.1 (Second Variation)

Let $\tilde{\alpha}: U \to \Omega$ be a 2-parameter variation of the geodesic γ with $W_i = \frac{\partial \tilde{\alpha}}{\partial u_i}(0,0) \in T_\gamma \Omega$

then $\frac{1}{2}\frac{\partial^2 E}{\partial u_1 \partial u_2}(0,0) = -\sum_t \langle W_2(t), \Delta_t \frac{DW_1}{dt} \rangle - \int_0^1 \langle W_2, \frac{D^2 W_1}{dt^2} + R(V, W_1)V \rangle dt$

$V = \frac{d\gamma}{dt}$. $\Delta_t \frac{DW_1}{dt} = \frac{DW_1}{dt}(t^+) - \frac{DW_1}{dt}(t^-)$

Pf. $\frac{1}{2}\frac{\partial E}{\partial u_2} = -\sum_t \langle \frac{\partial \alpha}{\partial u_2}, \Delta_t \frac{\partial \alpha}{\partial t} \rangle - \int_0^1 \langle \frac{\partial \alpha}{\partial u_2}, \frac{D}{\partial t}\frac{\partial \alpha}{\partial t} \rangle dt$

$\Rightarrow \frac{1}{2}\frac{\partial^2 E}{\partial u_1 \partial u_2} = -\sum_t \langle \underbrace{\frac{D}{\partial u_1}\frac{\partial \alpha}{\partial u_2}, \Delta_t \frac{\partial \alpha}{\partial t}}_{} \rangle - \sum_t \langle \frac{\partial \alpha}{\partial u_2}, \frac{D}{\partial u_1}\Delta_t \frac{\partial \alpha}{\partial t} \rangle$

[$\gamma = \bar{\alpha}(0,0)$ is an unbroken geodesic]

$- \int_0^1 \langle \underbrace{\frac{D}{\partial u_1}\frac{\partial \alpha}{\partial u_2}, \overset{0}{\frac{D}{\partial t}\frac{\partial \alpha}{\partial t}}}_{0} \rangle dt - \int_0^1 \langle \frac{\partial \alpha}{\partial u_2}, \frac{D}{\partial u_1}\frac{D}{\partial t}\frac{\partial \alpha}{\partial t} \rangle dt$

$= -\sum \langle W_2, \Delta_t \frac{D}{dt} W_1 \rangle - \int_0^1 \langle W_2, \frac{D}{\partial u_1}\frac{D}{\partial t} V \rangle dt$

Since $R(V, W_1)V = R(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial u_1})V$

$= \frac{D}{\partial u_1}\frac{D}{\partial t}V - \frac{D}{\partial t}\frac{D}{\partial u_1}V$

and $\frac{D}{\partial u_1}V = \frac{D}{\partial t}\frac{\partial \alpha}{\partial u_1} = \frac{D}{\partial t}W_1 \Rightarrow \frac{D}{\partial u_1}\frac{D}{\partial t}V = \frac{D^2 W_1}{dt^2} + R(V, W_1)V$

#

Cor 12.2 $E_{**}(W_1, W_2) = \frac{\partial^2 E}{\partial u_1 \partial u_2}(0,0)$ is a well-defined symmetric bilinear function of W_1, W_2.

Lemma 12.3 If γ is a minimal geodesic from p to q, then E_{**} is positive semi-definite.

Hence the index λ of E_{**} is zero.

13 Jacobi Fields: Null Space of E_{**}

A vector field J along a geodesic γ is a Jacobi field if $\frac{D^2 J}{dt^2} + R(V,J)V = 0$. $\left(V = \frac{d\gamma}{dt}\right)$

Def p and q are conjugate along γ if $\exists$ a non-zero Jacobi field J s.t. $J(a) = J(b) = 0$

The multiplicity is equal to $\dim\{J \text{ a Jacobi field} \mid J(a) = J(b) = 0\}$

The null space of E_{**} is the vector space

$N = \{W_1 \in T_\gamma \Omega \mid E_{**}(W_1, W_2) = 0 \quad \forall W_2\}$.

$\nu = \dim N$. $\quad E_{**}$ is degenerate if $\nu > 0$

Thm 13.1. $W_1 \in N$ iff W_1 is a Jacobi field.

Thus E_{**} is degenerate iff p, q are conjugate.

ν is the multiplicity of p, q

pf. Omitted here. Just use Thm 12.1 #.

Rmk 13.2 Actually $0 \le \nu < n$

<u>Ex</u> If M has constant zero curvature.

Then $\frac{D^2 J}{dt^2}=0$. Set $J(t)=\sum f^i P_i \Rightarrow \frac{d^2 f^i}{dt^2}=0$

Then a Jacobi field can have at most one zero.

Thus E** is non-degenerate.

Let α be a 1-parameter variation of γ, not necessarily keeping endpoints, s.t. $\bar{\alpha}(u)$ is a geodesic.

Lemma 13.3 If α is such a variation, then

$W(t)=\frac{\partial\alpha}{\partial u}(0,t)$ is a Jacobi field along γ.

pf. $\frac{D}{dt}\frac{\partial\alpha}{\partial t}=0 \Rightarrow 0=\frac{D}{\partial u}\frac{D}{\partial t}\frac{\partial\alpha}{\partial t}$

$$=\frac{D}{\partial t}\frac{D}{\partial u}\frac{\partial\alpha}{\partial t}+R\left(\frac{\partial\alpha}{\partial t},\frac{\partial\alpha}{\partial u}\right)\frac{\partial\alpha}{\partial t}$$

$$=\frac{D^2}{\partial t^2}\frac{\partial\alpha}{\partial u}+R\left(\frac{\partial\alpha}{\partial t},\frac{\partial\alpha}{\partial u}\right)\frac{\partial\alpha}{\partial t} \quad \#$$

Lemma 13.4 Every Jacobi field along a geodesic $\gamma:[0,1]\to M$ may be obtained by a variation of γ through geodesics

14. The Index Theorem

Thm 14.1 (Morse) The index of E_{**} is equal to the number of points $\gamma(t)$ $(0<t<1)$ that is conjugate to $\gamma(0)$ counted with its multiplicity

Cor 14.2. A geodesic $\gamma:[0,1]\to M$ can contain only finitely many conjugate points.

$\forall \gamma(t)$ is contained in an open set U s.t. $\forall p,q\in U$ are joined by a unique minimal geodesic which depends differentiably on the endpoints.

Choose a subdivision $0=t_0<\cdots<t_k=1$, s.t. $\gamma[t_{i-1},t_i]$ lies in such U, and $\gamma|_{[t_{i-1},t_i]}$ is minimal.

$T_\gamma\Omega(t_0,\cdots t_k)\subset T_\gamma\Omega$ consists of W s.t.

(i) $W|_{[t_{i-1},t_i]}$ is a Jacobi field along $\gamma|_{[t_{i-1},t_i]}$

(ii) $W(0)=W(1)=0$

$T'\subset T_\gamma\Omega$ consists of $W\in T_\gamma\Omega$ s.t. $W(t_i)=0$ $(i=0,\cdots k)$

Lemma 14.3 $T_\gamma\Omega = T'\oplus T_\gamma\Omega(t_0,\cdots t_k)$

These two spaces are perpendicular w.r.t. inner product E^{**}. $E^{**}|_{T'}$ is positive definite

Pf. $\forall w \in T_\gamma\Omega$, let w_1 be the unique vector field in $T_\gamma\Omega(t_0,\cdots t_k)$ s.t. $w_1(t_i)=w(t_i)$

Then $w-w_1\in T'$. And $T_\gamma\Omega(t_0,\cdots t_k)\cap T'=0$

Thus $T_\gamma\Omega=T_\gamma\Omega(t_0,\cdots t_k)\oplus T'$.

$w_1\in T_\gamma\Omega(t_0,\cdots t_k)$ $w_2\in T'$, then

$$\frac{1}{2}E_{**}(w_1,w_2)=-\sum_t\langle w_2(t),\Delta_t\frac{Dw_1}{dt}\rangle-\int_0^1\langle w_2,0\rangle dt=0.$$

$w\in T'$, one can check $E_{**}(w,w)\geq 0$.

If $E_{**}(w,w)=0$, then $\forall w_2\in T'$.

$$0\leq E_{**}(w+cw_2,w+cw_2)=2c\,E_{**}(w_2,w)+c^2E_{**}(w_2,w_2)$$

c is arbitrary $\Rightarrow$ $E_{**}(w_2,w)=0$

$\forall w_1\in T_\gamma\Omega(t_0,\cdots t_k)$ $E_{**}(w_1,w)=0$ $\Rightarrow w\in N$

But $N=\{\text{Jacobi fields}\}$ $\Rightarrow w=0$. #

Lemma 14.4 The index (nullity) of E_{**} is equal to the index (nullity) of E_{**} restricted to the space $T_\gamma\Omega(t_0,\cdots t_k)$.

Now focus on the proof of Thm 14.1

$\gamma_\tau\overset{\Delta}{=}\gamma|_{[0,\tau]}$ $\lambda(\tau)=$ the index of $(E_0^\tau)_{**}$

We are going to compute $\lambda(1)$.

Step 1. $\lambda(\tau)$ is a monotone function of τ

Step 2. $\lambda(\tau)=0$ for small values of τ.

[Trivial]

Step 3. For sufficiently small $\varepsilon>0$, $\lambda(\tau-\varepsilon)=\lambda(\tau)$

[pf. Assume the subdivision is chosen, s.t. $t_i<\tau<t_{i+1}$. Then $\lambda(\tau)$ is the index of the quadratic fun H_τ on the space of broken Jacobi fields along γ_τ, which is isomorphic to

$$\Sigma = T_{\gamma(t_1)}M \oplus T_{\gamma(t_2)}M \cdots \oplus T_{\gamma}M(t_i),$$

which is independent of τ.

H_τ is negative definite on $V\subset\Sigma$ of dim $\lambda(\tau)$

For τ' close to τ, $H_{\tau'}$ is negative definite on V.

Thus $\lambda(\tau')\geq\lambda(\tau)$. By Step 1, $\lambda(\tau-\varepsilon)\leq\lambda(\tau)$ $\Rightarrow \lambda(\tau-\varepsilon)=\lambda(\tau)$].

Step 4 Let ν be the nullity of $(E_0^\tau)_{**}$, then for small $\varepsilon>0$, $\lambda(\tau+\varepsilon)=\lambda(\tau)+\nu$.

[pf. Omitted here].

Thus Thm 14.1 is a result of Step 2 & 4

#.

15. A Finite Dimensional Approximation to Ω^c

Let ρ be the topological metric on M coming from the Riemannian metric.

Given $\omega, \omega' \in \Omega$ with arc-lengths $s(t), s'(t)$, define the distance

$$d(\omega, \omega') = \max_{0 \le t \le 1} \rho(\omega(t), \omega'(t)) + \sqrt{\int_0^1 \left(\frac{ds}{dt} - \frac{ds'}{dt}\right)^2 dt}$$

This metric gives a topology on Ω.

Given $c > 0$, denote $\Omega^c = E^{-1}([0,c]) \subset \Omega$

$$\operatorname{Int} \Omega^c = E^{-1}([0,c))$$

$0 = t_0 < t_1 \cdots < t_k = 1$ $\quad \Omega(t_0 \cdots t_k)$ consists of ω s.t.

(1) $\omega(0) = p \qquad \omega(1) = q$

(2) $\omega|_{[t_{i-1}, t_i]}$ is a geodesic

Then define $\Omega(t_0 \cdots t_k)^c = \Omega^c \cap \Omega(t_0 \cdots t_k)$

$$\operatorname{Int} \Omega(t_0 \cdots t_k)^c = (\operatorname{Int} \Omega^c) \cap \Omega(t_0 \cdots t_k)$$

Lemma 15.1. Let M be a complete Riemannian manifold and c fixed with $\Omega^c \neq \phi$. Then for all sufficiently fine subdivisions $(t_0, \cdots t_k)$, $\operatorname{Int} \Omega(t_0 \cdots t_k)^c$ can be given the structure of a smooth finite dimensional manifold.

Pf. $S = \{x \in M \mid \rho(x,p) \le \sqrt{c}\}$

Then every $\omega \in \Omega^c$ lies in $S \subset M$

M is complete $\Rightarrow S$ is compact

Then $\exists \varepsilon > 0$ s.t. $\forall x, y \in S$, $\rho(x,y) < \varepsilon$, $\exists$ a unique geodesic connecting x, y with length $< \varepsilon$.

Choose $(t_0, \cdots t_k)$ s.t. $t_i - t_{i-1} < \frac{\varepsilon^2}{c}$

Then for $\omega \in \Omega(t_0, \cdots t_k)^c$,

$$(L_{t_{i-1}}^{t_i} \omega)^2 = (t_i - t_{i-1})(E_{t_{i-1}}^{t_i} \omega) \le \varepsilon^2.$$

$\Rightarrow \omega$ is determined by $\omega(t_1), \cdots \omega(t_{k-1})$

Then $\text{Int}\,\Omega(t_0, \cdots t_k)^c \cong$ a certain open subset of M^{k-1}.

Then take the differentiable structure of M^{k-1} #

Denote this manifold by B. Let $E' = E|_B$.

Thm 15.2 $E' : B \to \mathbb{R}$ is smooth. $\forall a < c$, $B^a = (E')^{-1}[0,a]$ is compact, and is a deformation retract of Ω^a

Critical points of E' = Critical points of E in $\text{Int}\,\Omega^c$ namely the unbroken geodesic from p to q with length $< \sqrt{c}$

By Thm 15.2 and Thm 2.4. we have

Thm 15.3. M complete. p, q not conjugate along any geodesic of length $\le \sqrt{a}$. Then Ω^a has a homotopy type of a finite CW-complex with one cell of dim λ for each geodesic in Ω^a which $E_{**} = \lambda$

pf. of Thm 15.2

$\omega \in B$ depends on $\omega(t_1) \cdots \omega(t_{k-1}) \in M^{k-1}$

$$E'(\omega) = \sum_{i=1}^{k} \frac{\rho^2(\omega(t_{i-1}), \omega(t_i))}{t_i - t_{i-1}}$$

For $a < c$, $B^a \underline{n} \{(p_1 \cdots p_{k-1}) \in S \mid \sum_{i=1}^{k} \frac{\rho(p_{i-1}, p_i)^2}{t_i - t_{i-1}} \le a\}$, closed $\Rightarrow$ compact

Let $r(\omega)$ denote the unique broken geodesic in B s.t. $r(\omega)|_{[t_{i-1}, t_i]}$ is a geodesic of length $< \varepsilon$ from $\omega(t_{i-1})$ to $\omega(t_i)$

$$\rho(p, \omega(t))^2 \le (L\omega)^2 \le E\omega < c \Rightarrow \omega[0,1] \subset S$$

Since $\rho(\omega(t_{i-1}), \omega(t_i))^2 \le (t_i - t_{i-1})(E_{t_{i-1}}^{t_{i+1}} \omega) < \varepsilon^2$

$r(\omega)$ can be defined

Let $r_u : \operatorname{Int} \Omega^c \to \operatorname{Int} \Omega^c$, for $u \in [t_{i-1}, t_i]$.

$$\begin{cases} r_u(\omega)|[0, t_{i-1}] = r(\omega)|[0, t_{i-1}] \\ r_u(\omega)|[t_{i-1}, u] = \text{minimal geodesic from } \omega(t_{i-1}) \text{ to } \omega(u) \\ r_u(\omega)|[u, 1] = \omega|[u, 1] \end{cases}$$

$\{r_u\}$ is a deformation retract from $\operatorname{Int} \Omega^c$ to B. Similar for $a < c$

#

16. The Topology of the Full Path Space

$\Omega^* = \{\omega : [0,1] \to M$, from p to q, continue in the compact open topology$\}$

This topology is also induced by $d^*(\omega, \omega') = \max_t \rho(\omega(t), \omega'(t))$

We've introduced $\Omega = \{$piecewise C^∞ paths$\}$ with

$$d(\omega, \omega') = d^*(\omega, \omega') + \sqrt{\int_0^1 \left(\frac{ds}{dt} - \frac{ds'}{dt}\right)^2 dt}$$

$d \geq d^* \Rightarrow i : \Omega \to \Omega^*$ is continuous.

Thm 16.1 i is a homotopy equivalent

[pf omitted here]

It's known that Ω^* has the homotopy type of a CW-complex. thus

Cor 16.2 Ω has the homotopy type of a CW-complex.

Thm 16.3 (Fundamental Theorem of Morse Theory)

M complete, p, q not conjugate along any geodesic.

Then $\Omega(M, p, q)$ (or $\Omega^*(M, p, q)$) has the homotopy type of a countable CW-complex which contains one cell of dim λ for each geodesic from p to q of index λ

Pf. Choose $a_0 < a_1 < a_2 \cdots$ which are not critical

(a_{i-1}, a_i) contains only one critical value of E.

Consider $\Omega^{a_0} \subset \Omega^{a_1} \cdots$, assume $\Omega^{a_0} = \phi$

Then by Thm 15.2, each Ω^{a_i} has the homotopy type of $\Omega^{a_{i-1}}$ with some cells attached. one λ-cell for each geodesic of index λ in $E^{-1}(a_{i-1}, a_i)$.

One can construct $K_0 \subset K_1 \subset \cdots$ of CW-complexes with cells required, and

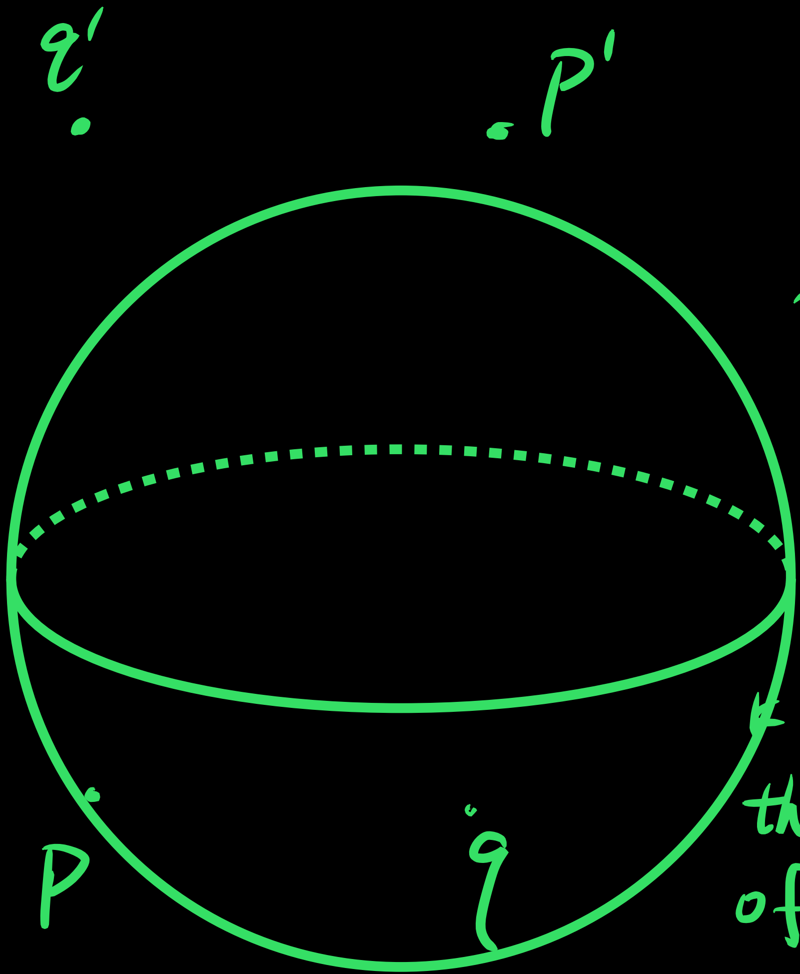
$$\begin{array}{ccccccc} \Omega^{a_0} & \subset & \Omega^{a_1} & \subset & \Omega^{a_2} & \subset & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ K_0 & \subset & K_1 & \subset & K_2 & \subset & \cdots \end{array}$$

$f : \Omega \to K$ be the direct limit mapping. f induces isomorphisms of homotopy groups in all dimension.

Since Ω has the homotopy type of a CW-complex. f is a homotopy equivalence.

#.

Ex: $\Omega(S^n)$. Suppose p, q non-conjugate. p' is the antipode of p



γ_0: short great circle
γ_1: long great circle
γ_2: $pqp'q'pq$

k: the number of times that p occurs in the interior of γ_k.

$\gamma(\gamma_k) = \mu_1 + \cdots \mu_k = k(n-1)$, since each of the p, p' in the interior is conjugate to p with multiplicity $n-1$

Cor 16.4 $\Omega(S^n)$ has the homotopy type of a CW-complex with one cell in the dimension $k(n-1)$ $(k \geq 0)$

Since $\Omega(S^n)$ has non-trivial homology in infinitely-many dimension.

Cor 16.5 M has the homotopy type of S^n $(n \geq 2)$, then any two non-conjugate points of M are joined by infinitely many geodesics

17 Existence of Non-conjugate Points

$f: N \to M$ is critical at $x \in N$ if
$df: T_x N \to T_{f(x)} M$ is not 1-1

Thm 17.1 $\exp_p v$ is conjugate to p along the geodesic γ_v from p to $\exp_p v$ iff $\exp_p$ is critical at v

pf. $\exp_p$ is critical at v. Then $\exists X \in T_v(T_pM)$ s.t. $\exp_{p*}(X) = 0$.

Let $u \to v(u)$ be a path in T_pM s.t. $v(0) = v$. $\frac{dv}{du}(0) = X$.

Then $\alpha(u,t) = \exp_p(tv(u))$ is a variation through geodesic γ_v given by $t \to \exp_p tv$.

Thus $W(t) = \frac{\partial}{\partial u}(\exp_p(tv(u)))|_{u=0}$ is a Jacobi field.

$W(0) = 0$ $\quad W(1) = \frac{\partial}{\partial u}(\exp_p(v(u)))|_{u=0} = \exp_{p*} \frac{dv(u)}{du}(0) = \exp_{p*} X = 0$

But $W \not\equiv 0 \Rightarrow p, \exp_p v$ are conjugate.

#

Cor 17.2 Let $p \in M$, then for almost all $q \in M$, p is not conjugate to q along any geodesic.

pf This is from Thm 17.1 and Sard's Theorem #

18 Some Relations between Topology and Curvature

Lemma 18.1 $\langle R(A,B)A, B\rangle \le 0$ for $\forall A, B \in T_pM$ $\forall p$.

then no two points of M are conjugate along any geodesic

Pf. γ a geodesic. $V = \frac{d\gamma}{dt}$ J a Jacobi field

then $\frac{D^2 J}{dt^2} + R(V,J)V = 0$

$\langle \frac{D^2 J}{dt^2}, J\rangle = -\langle R(V,J)V, J\rangle \ge 0$

$\frac{d}{dt}\langle \frac{DJ}{dt}, J\rangle = \langle \frac{D^2J}{dt^2}, J\rangle + \|\frac{DJ}{dt}\|^2 \ge 0$

If $J(0) = J(t_0) = 0$. ($t_0 > 0$) then $\langle \frac{DJ}{dt}, J\rangle$ is 0 at 0 and t_0

Then $\langle \frac{DJ}{dt}, J\rangle \equiv 0 \Rightarrow J(0) = \frac{DJ}{dt}(0) = 0 \Rightarrow J \equiv 0$. #

Thm 18.2 M is simply connected, complete. $\langle R(A,B)A, B\rangle \le 0$

Then any two points of M are joined by a unique geodesic. Furthermore, M is diffeomorphic to $\mathbb{R}^n$.

pf By 18.1, no points are conjugate. Thus every geodesic from p to q has index $\lambda = 0$

Then by Thm 16.3 $\Omega(M;p,q)$ has the homotopy type of a 0-dim CW-complex, with one vertex for each geodesic.

M is simply connected $\Rightarrow \Omega(M;p,q)$ is connected.

Since there is only one geodesic from p to q

Then $\exp_p$ is invertible and non-critical $\Rightarrow M \cong T_pM \cong \mathbb{R}^n$ #

Cor 18.3 M is complete $\langle R(A,B)A,B\rangle \le 0$.
then $\pi_i(M)$ for $i>1$ $\pi_1(M)$ contains no element of finite order other than 1

Def. The Ricci tensor at p is a bilinear pairing

$$K: T_pM \times T_pM \to \mathbb{R}.$$

$K(U_1,U_2)$ is the trace of the linear transform

$$W \mapsto R(U_1,W)U_2$$

K is symmetric.

Let $U_1 \cdots U_n$ be an orthonormal basis for T_pM.

$$K(U_n,U_n) = \sum_{i=1}^{n-1} \langle R(U_n,U_i)U_n, U_i\rangle$$

Thm 18.4 (Myers) $K(U,U) \ge \frac{n-1}{r^2}$ for every $|U|=1$, $r>0$. Then every geodesic on M of length $>\pi r$ contains conjugate points. thus not minimal

Pf. $\gamma:[0,1]\to M$ with length L Choose parallel vector fields $P_1 \cdots P_n$ along γ which are orthonormal

Assume $V=\frac{d\gamma}{dt}=LP_n$, $\frac{DP_i}{dt}=0$.

Let $W_i(t)=(\sin\pi t)P_i(t)$

Then $\frac{1}{2}E_{**}(W_i,W_i) = -\int_0^1 \langle W_i, \frac{D^2W_i}{dt^2} + R(V,W_i)V \rangle dt$

$= \int_0^1 (\sin\pi t)^2 (\pi^2 - L^2 \langle R(P_n,P_i)P_n, P_i \rangle) dt$

$\Rightarrow \frac{1}{2}\sum_{i=1}^{n-1} E_{**}(W_i,W_i) = \int_0^1 (\sin\pi t)^2 ((n-1)\pi^2 - L^2 K(P_n,P_n)) dt < 0$

then $\exists i$ s.t. $E_{**}(W_i,W_i) < 0 \Rightarrow$ index of $\gamma > 0$

#

Cor 18.5 M complete. $K(U,U) \geq \frac{n-1}{r^2}$ for all $|U|=1$

then M is compact, with diameter $\leq \pi r$.

Thm 18.6 M is compact, K is everywhere positive definite. then $\Omega(M;p,q)$ has the homotopy type of a CW-complex having only finitely many cells in each dimension.

Pf. Since $\{U \mid |U|=1\}$ is compact, $K(U,U)$ attains a minimum, denoted by $\frac{n-1}{r^2} > 0$. Then every geodesic $\gamma \in \Omega(M;p,q)$ of length $> \pi r$ has index $\lambda \geq 1$

For geodesic γ of length $> k\pi r$,

$\forall i = 1,\cdots k$, $\exists X_i$ along γ which vanishes outside $(\frac{i-1}{k}, \frac{i}{k})$

and $E_{**}(X_i,X_i) < 0$

Clearly $E_{**}(X_i,X_j) = 0$ for $i \neq j$. then $\langle X_1 \cdots X_k \rangle$ span a k-dim subspace where E_{**} is negative definite

Thus γ has the index $\lambda \geq k$.

By Thm 15.3, 16.3, the result follows

#

RART IV.

Applications to Lie Groups and Symmetric Spaces

19. Symmetric Spaces

A symmetric space is a connected Riemannian manifold M. s.t.

$\forall p$. $\exists$ isometry $I_p : M \to M$ fixes p, and

$\forall$ geodesic γ with $\gamma(0)=p$ $\quad I_p(\gamma(t)) = \gamma(-t)$

Lemma 19.1 γ a geodesic. $p=\gamma(0)$ $q=\gamma(c)$

Then $I_q I_p(\gamma(t)) = \gamma(t+2c)$ Moreover, $I_q I_p$ preserves all parallel vector fields along γ.

Pf. $\bar\gamma(t) = \gamma(t+c)$ $\quad \bar\gamma(0)=q$.

$$I_q I_p(\gamma(t)) = I_q(\gamma(-t)) = I_q(\bar\gamma(-t-c)) = \bar\gamma(t+c) = \gamma(t+2c).$$

If V is parallel then $I_{p*}(V)$ is parallel. [isometry]

$$I_{p*}V(0) = -V(0).$$

Thus $I_{p*}V(t) = -V(-t)$. $\Rightarrow I_{q*}I_{p*}(V(t)) = V(t+2c)$. #

Cor 19.2 M is complete since each γ can be extended indefinitely

Cor 19.3 I_p is unique since $\forall q$ can be joined to p by a geodesic.

Cor 19.4 U, V, W are parallel along γ. then

$R(U,V)W$ is parallel along γ

pf If X is parallel. note $\langle R(U,V)W, X\rangle$ is constant

along γ: Let $p=\gamma(0)$ $q=\gamma(c)$. $T=I_{\gamma(\frac{c}{2})} I_p$. $T(p)=q$.

then $\langle R(U_q, V_q)W_q, X_q\rangle = \langle R(T_* U_p, T_* V_p)T_* W_p, T_* X_p\rangle$

$= \langle R(U_p, V_p)W_p, X_p\rangle$

Then $\langle R(U,V), W\rangle$ is parallel. #.

[Manifolds satisfying 19.4 is called locally symmetric]

$\gamma: \mathbb{R} \to M$ a geodesic in a locally symmetric manifold.

$v = \frac{d\gamma}{dt}(0)$. $K_v: T_pM \to T_pM$ $e_1, \cdots e_n$: eigenvalue.

$w \mapsto R(v,w)v$.

Thm 19.5 The conjugate points to p along γ are

$\gamma(\frac{\pi k}{\sqrt{e_i}})$ $(k \in \mathbb{Z}^*)$ $(e_i > 0)$. The multiplicity of

$\gamma(t)$ is equal to number of e_i s.t. $\frac{t\sqrt{e_i}}{\pi} \in \mathbb{Z}$

pf. $\langle K_v(W), W'\rangle = \langle W, K_v(W')\rangle \Rightarrow \exists$ orthonormal $U_1 \cdots U_n$

s.t. $K_v(U_i) = e_i U_i$

Extend U_i along γ Since M is locally symmetric

$R(V, U_i)V = e_i U_i$ holds along γ.

$\forall W = \sum w_i U_i$. $\frac{D^2 W}{dt^2} + K_v(W) = 0 \Rightarrow \sum \frac{d^2 w_i}{dt^2} U_i + \sum e_i w_i U_i = 0$

$\Rightarrow w_i'' + e_i w_i = 0$

$e_i > 0 \Rightarrow w_i(t) = c_i \sin(\sqrt{e_i}\, t)$.

$e_i = 0 \Rightarrow w_i(t) = c_i t$

$e_i < 0 \Rightarrow w_i(t) = c_i \sinh(\sqrt{|e_i|}\, t)$. The result follows. #.

20. Lie Groups as Symmetric Spaces

Lemma 20.1 The geodesics γ in G with $\gamma(0)=e$ are precisely the one-parameter subgroups of G.

A vector field X on a Lie group G is called left invariant if $(L_a)_*(X_b)=X_{a\cdot b}$ $(\forall a,b\in G)$

If X,Y are left invariant, so is $[X,Y]$

The Lie algebra $\mathfrak{g}$ of G is the vector space of all left invariant vector fields with $[\cdot,\cdot]$.

Thm 20.2 G is a Lie group with a left and right invariant Riemannian metric

If X,Y,Z,W are left invariant vector fields, then

(a) $\langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle$

(b) $R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$

(c) $\langle R(X,Y)Z, W\rangle = \frac{1}{4}\langle [X,Y],[Z,W]\rangle$

pf. Since the integral curves of X are left translates of 1-parameter subgroups, therefore geodesics.

we have $\nabla_X X=0$

Thus $\nabla_{X+Y} X+Y = \nabla_X X+\nabla_Y Y+\nabla_X Y+\nabla_Y X = 0$

$\Rightarrow \nabla_X Y+\nabla_Y X=0.$

But $\nabla_X Y - \nabla_Y X = [X,Y] \Rightarrow [X,Y] = 2\nabla_X Y$

$Y\langle X,Z\rangle = \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle$
$\overset{\parallel}{0}$

$\Rightarrow \langle [Y,X], Z\rangle + \langle X, [Y,Z]\rangle = 0$

$\Rightarrow \langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle$

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$
$$= -\frac{1}{4}[X,[Y,Z]] + \frac{1}{4}[Y,[X,Z]] + \frac{1}{2}[[X,Y],Z]$$
$$\overset{\text{Jacobi}}{=} \frac{1}{4}[[X,Y],Z]$$

#

Cor 20.3 Sectional curvature $\langle R(X,Y)X,Y\rangle = \frac{1}{4}\langle [X,Y],[X,Y]\rangle \geq 0$

Equality holds iff $[X,Y]=0$.

The center c of g is $\{X\in g \mid \forall Y\in g \ [X,Y]=0\}$

Cor 20.4 If G has a left and right invariant metric and if the Lie algebra g has trivial center, then G is compact with finite fundamental group.

pf. $X_1 \in g$ is a unit vector. Extend to an orthonormal basis $X_1 \cdots X_n$. $K(X_1,X_1) = \sum_{i=1}^{n} \langle R(X_1,X_i)X_1,X_i\rangle > 0$

$K(X_1,X_1)$ is bounded away from 0 since the unit sphere in g is compact.

Then by Cor 18.5 the result follows. #

Cor 20.5 A simply connected Lie group G with left and right invariant metric splits as a Cartesian product $G' \times \mathbb{R}^k$, where G' is compact and $\mathbb{R}^k$ is the additive Lie group of Euclidean space. Furthermore, G' has trivial center.

Thm 20.6 (Bott) G is compact, simply connected Lie group. Then the loop space $\Omega(G)$ has the homotopy type of a CW-complex with no odd-dim cells, and with only finitely many λ-cells for each even value λ.

Pf. Choose p, q in G which are not conjugate.

then $\Omega(G; p, q)$ has the homotopy type of a CW-complex with one cell of dim λ for each geodesic from p to q of index λ (finitely many). It remains to show λ is even.

$\gamma(0) = p$. $v = \frac{d\gamma}{dt}(0) \in T_pG \cong \mathfrak{g}$.

The conjugate points of p are determined by the eigenvalues of K_v $T_pG \to T_pG$

$$w \mapsto R(v,w)v = \tfrac{1}{4}[[v,w],v]$$

$$\Rightarrow K_v = -\tfrac{1}{4}(\mathrm{Ad}\, v) \circ (\mathrm{Ad}\, v)$$

Define $\mathrm{Ad}\, v : \mathfrak{g} \to \mathfrak{g}$ $\quad w \mapsto [v,w]$

Ad v is skew-symmetric.

$$\langle \mathrm{Ad}\, v(w), w' \rangle = -\langle \mathrm{Ad}\, v(w'), w \rangle$$

Then $\exists$ an orthonormal bais of G s.t the matrix of Ad v is

$$\begin{pmatrix} 0 & a_1 & & \\ -a_1 & 0 & a_2 & \\ & -a_2 & \ddots & \\ & & & 0 \end{pmatrix}$$

Then $(\mathrm{Ad}\, v) \circ (\mathrm{Ad}\, v)$ has the matrix

$$\begin{pmatrix} -a_1^2 & & & \\ & -a_1^2 & & \\ & & -a_2^2 & \\ & & & \ddots \end{pmatrix}.$$

Thus the non-zero eigenvalues of $K_v = -\frac{1}{4}(\mathrm{Ad}\, v)^2$ are positive, and in pairs

Thus the conjugate points have even multiplicity.

Then the result comes from Index Theorem

#.

21. Whole Manifolds of Minimal Geodesics

M complete. $p, q \in M$ $\rho(p,q) = \sqrt{d}$.

Thm 21.1 In the space Ω^d of minimal geodesics from p to q is a topological manifold, and if every non-minimal geodesic from p to q has index $\geq \lambda_0$. Then the relative homotopy group $\pi_i(\Omega, \Omega^d) = 0$ $(0 \leq i < \lambda_0)$.

Then $i: \pi_i(\Omega^d) \to \pi_i(\Omega)$ is an isomorphism $(i \leq \lambda_0 - 2)$.

It's also known that $\pi_i(\Omega)$ is isomorphic to $\pi_{i+1}(M)$

Cor 21.2 With the same hypotheses, $\pi_i(\Omega^d)$ is isomorphic to $\pi_{i+1}(M)$ for $0 \leq i \leq \lambda_0 - 2$.

For antipodal points on S^{n+1}, clearly $\lambda_0 \geq 2n$

Cor 21.3 (Freudenthal suspension Thm)
$\pi_i(S^n)$ is isomorphic to $\pi_{i+1}(S^{n+1})$ $(i \leq 2n-2)$

Lemma 21.4 $K \subset \mathbb{R}^n$ compact, U a nbhd of K. $f: U \to \mathbb{R}$ is smooth. s.t. all critical points of f in K have index $\geq \lambda_0$.

If $g: U \to \mathbb{R}$ is smooth and "close" to f, i.e.

$$\left|\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\right| < \varepsilon, \quad \left|\frac{\partial^2 g}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_i \partial x_j}\right| < \varepsilon \quad (\forall i, j)$$

uniformly in K, for some small ε.

Then all critical points of g in K have index $\geq \lambda_0$

Pf. Omitted here. #

Lemma 21.5. $f \in C^\infty(M)$ with minimum 0, s.t. $M^c = f^{-1}([0, c])$ is compact. If M^0 is a manifold and every critical point in $M - M^0$ has index $\geq \lambda_0$, then $\pi_r(M, M^0) = 0$ $(0 \leq r < \lambda_0)$

Pf. M^0 is a retract of some nbhd $U \subset M$

Replace U by a smaller nbhd, assume that each point of U is joined to the corresponding point of M^0 by a unique minimal geodesic.

Thus U can be deformed into M^0

I^r is the unit cube of dim $r < \lambda_0$

$h: (I^r, \dot{I}^r) \to (M, M^0)$ be any map.

$c = \max\limits_{p \in h(I^r)} f(p) \qquad \delta = \frac{1}{3} \min\limits_{p \in M - U} f(p)$

Choose $g \in C^\infty(M^{c+2\delta})$ approximates f closely with no degenerate critical points:

(i) $|f(x)-g(x)|<\delta \quad \forall x \in M^{c+2\delta}$

(ii) Index of g at each critical point in $f^{-1}([\delta, c+2\delta])$ is $\geq \lambda_0$

[g exists by Lemma 21.4]

g is smooth on the compact region $g^{-1}([2\delta, c+\delta]) \subset f^{-1}([\delta, c+2\delta])$

thus $g^{-1}(-\infty, c+\delta]$ has the homotopy type of $g^{-1}(-\infty, 2\delta]$ with cells of dim $\geq \lambda_0$ attached.

Consider $h: I^r, \dot{I}^r \to M \cap g^{-1}(-\infty, c+\delta], M^0$

$r<\lambda_0$. thus h is homotopic with $g^{-1}(-\infty, c+\delta]$. M^0 to

$$h': I^r, \dot{I}^r \to g^{-1}(-\infty, 2\delta] \; M^0 \subset (U, M^0)$$

But U can be deformed into M^0 with M. thus h' is homotopic within (M, M^0) to $h'': I^r, \dot{I}^r \to M^0, M^0$. #

pf of 21.1. It's sufficient to prove $\pi_i(\Omega^c, \Omega^d)=0$ for any large c. By §15. Ω^c contains a smooth manifold $\Omega^c(t_0 \cdots t_k)$ as deformation retract.
$\supset \Omega^d$

Let $F: [d, c) \to [0, \infty)$ be any diffeomorphism then

$F \circ E: \Omega^c(t_0, \cdots t_k) \to \mathbb{R}$ satisfies the hypothesis of 21.5.

$\Rightarrow \pi_i(\Omega^c(t_0, \cdots t_k), \Omega^d) \cong \pi_i(\Omega^c, \Omega^d)=0 \quad (i<\lambda_0)$ #

22. The Bott Periodicity Theorem for the Unitary Group

$U(n) = \{ S \in \mathbb{C}^{n\times n} \mid SS^* = I_n \}$ is a smooth submanifold

$T_I U(n)$ can be identified with the space of

$n\times n$ skew-Hermitian space

$\mathfrak{g}$ (Lie algebra)

$A, B \in \mathfrak{g}$, define $\langle A, B \rangle = tr(AB^*)$ It induces a left and right invariant metric

$SU(n) = \{ S \in U(n) \mid \det S = 1 \}$

$\mathfrak{g}' = T_I SU(n) = \{ A \in \mathbb{C}^{n\times n} \mid A + A^* = 0,\ trA = 0 \}$

Consider the set of all geodesics in $U(n)$ from I to $-I$. i.e. $A \in T_I U(n) = \mathfrak{g}$, $\exp A = -I$

WLOG, assume A is in the diagonal form.

$$A = \begin{pmatrix} ia_1 & & \\ & \ddots & \\ & & ia_n \end{pmatrix} \qquad \exp A = \begin{pmatrix} e^{ia_1} & & \\ & \ddots & \\ & & e^{ia_n} \end{pmatrix}$$

$$\exp A = -I \Rightarrow A = \begin{pmatrix} k_1 i\pi & & \\ & \ddots & \\ & & k_n i\pi \end{pmatrix}. \quad k_i \text{ is odd.}$$

The length of the geodesic $t \mapsto \exp tA$ is

$$|A| = \sqrt{trAA^*} = \pi\sqrt{k_1^2 + \cdots + k_n^2}$$

Thus A determines a minimal geodesic iff $k_i = \pm 1$

Then the length is $\pi\sqrt{n}$.

Regarding $A \in L_n(\mathbb{C}^n)$, it is completely determined by specifying $\text{Eigen}(i\pi) = \{v \in \mathbb{C}^n \mid Av = i\pi v\}$ and $\text{Eigen}(-i\pi)$.

Since $\mathbb{C}^n = \text{Eigen}(i\pi) \oplus \text{Eigen}(-i\pi)$, A is determined by $\text{Eigen}(i\pi)$, an arbitrary subspace of $\mathbb{C}^n$

Replace $U(n)$ by $SU(n)$, let $n = 2m$, then $a_1 + \cdots a_{2m} = 0$.

Thus $\text{Eigen}(i\pi)$ is of dim m, then we have

Lemma 22.1 The space of minimal geodesics from I to $-I$ in $SU(2m)$ is homeomorphic to the complex Grassmann manifold $G_m(\mathbb{C}^{2m}) = \{V \subset \mathbb{C}^{2m} \mid V \text{ is a } m\text{-dim subspace}\}$.

Lemma 22.2 Every non-minimal geodesic from I to $-I$ in $SU(2m)$ has index $\geq 2m+2$.

[Pf omitted here. #]

Thus we have

Thm 22.3 (Bott). The inclusion map $G_m(\mathbb{C}^{2m}) \to \Omega(SU(2m); I, -I)$ induces isomorphisms

$$\pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i+1} SU(2m) \quad (i \leq 2m).$$

Lemma 22.4 $\pi_i Gr_n(\mathbb{C}^{2n}) \simeq \pi_{i-1} U(n)$ $(i \le 2n)$.

Furthermore $\pi_{i-1} U(n) \simeq \pi_{i-1} U(n+1) \simeq \pi_{i-1} U(n+2) \cdots$ $(i \le 2n)$

✓ $\pi_j U(n) \simeq \pi_j SU(n)$ $(j \ne 1)$

(i-1)st stable homotopy group $\pi_{i-1} U$

Periodicity Thm. $\pi_{i-1} U \simeq \pi_{i+1} U$ $(i \ge 1)$.

pf. $\pi_{i-1} U = \pi_{i-1} U(n) \simeq \pi_i Gr_n(\mathbb{C}^{2n}) \simeq \pi_{i+1} SU(2n) \simeq \pi_{i+1} U$. #.

Thm 22.5 (Bott). The stable homotopy group $\pi_i U$ of the unitary groups are periodic with period 2.

In fact,

$\pi_0 U \simeq \pi_2 U \simeq \pi_4 U \simeq \cdots$ are 0.

$\pi_1 U \simeq \pi_3 U \simeq \pi_5 U \cdots$ are infinite cyclic.